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Thermoelastic vibrations of a transversely isotropic plate with thermal relaxations

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Abstract

Propagation of plane harmonic thermoelastic waves in a thin, flat, infinite homogeneous, transversely isotropic plate of finite width is studied, in the context of generalized theory of thermoelasticity. Green and Lindsay (GL) theory, in which, thermal and thermo-mechanical relaxations are governed by two different time constants, is selected for the study. The frequency equations corresponding to the symmetric and antisymmetric modes of vibration of the plate are obtained, and some limiting and special cases of the frequency equations are then discussed. The results have been verified numerically and are represented graphically. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Generalized thermoelasticity; Thermal relaxations; Vibrations; Frequency equation

1. Introduction

The theory to include the effect of temperature change, known as the theory of thermoelasticity, has been well established. According to the theory, the temperature field is coupled with the elastic strain field. The classical theory of thermoelasticity predicts infinite speed of transportation, which contradicts the physical facts. Lord and Shulman (1967) (referred to as the LS theory) and Green and Lindsay (1972) (referred to as the GL theory) extended the coupled theory of thermoelasticity by introducing the thermal relaxation time in the constitutive equations. This new theory, which eliminates the paradox of infinite velocity of heat propagation, is called generalized theory of thermoelasticity. This generalized thermoelasticity theory that admits finite speed for the propagation of thermoelastic disturbances has received much attention in recent years. The works of Chandrasekharaiah (1986), Ignaczak (1989), Green and Naghdi (1991, 1992), and Hetnarski and Ignaczak (1994) contain more detailed discussions on this phenomenon. The LS model introduces a single time constant to dictate the relaxation of thermal propagation, as well as the rate of change of strain rate and the rate of change of heat generation. In the GL theory, on the other hand, the thermal and thermo-mechanical relaxations are governed by two different time constants.

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Deresiewicz (1975) considered the propagation of waves in thermoelastic plates under plain strain state. The propagation of thermoelastic waves in a plate under plane stress by using generalized theories of thermoelasticity has been studied by Chandrasekheraiah and Srinantiah (1984, 1985), Massalas (1986). Here, we mention that several authors Puri (1973, 1975), Agarwal (1978, 1979), Agarwal (1979), Tao and Prevost (1984), Massalas and Kalpakidis (1987b), and Daimaruya and Naitoh (1987) have considered the propagation of generalized thermoelastic waves in plates of isotropic media. Massalas and Kalpakidis (1987a) used the generalized theory of Lord and Shulman to study the characteristics of wave motion in a thin plate under plane stress state with mixed boundary conditions. They used Lamé's potentials to derive the frequency equation. Verma and Hasebe (1999) studied the propagation of generalized thermoelastic vibrations in infinite plates in the context of generalized thermoelasticity.

Banerjee and Pao (1974) extended this theory to anisotropic heat conducting elastic materials. Dhaliwal and Sherief (1980) treated the problem in more systematic manner. They derived governing field equations of generalized thermoelastic media and proved that these equations have a unique solution. Sharma and Sidhu (1986) discussed the propagation of plane harmonic waves in a homogeneous anisotropic generalized thermoelastic solid. Chadwick and Seet (1970), and Chadwick (1979) investigated the thermoelastic wave propagation in transversely isotropic and homogeneous anisotropic heat conducting elastic materials, respectively.

In this paper, we investigate the propagation of plane harmonic waves in an infinite homogeneous transversely isotropic plate of thickness $2d$ in the context of Green and Lindsay (1972) generalized theory of thermoelasticity. The frequency equations corresponding to the symmetric and antisymmetric thermoelastic modes of vibration are obtained and discussed for heat conducting thermoelastic plate, and some limiting cases of the frequency equations are then discussed. Relevant results of previous investigations are deduced as special cases. The results have been verified numerically and are represented graphically.

2. Formulation

We consider an infinite, homogeneous transversely isotropic, thermally conducting elastic plate at uniform temperature T_0 in the undisturbed state having thickness $2d$. Let the faces of the plate be the planes $z = \pm d$, referred to a rectangular set of Cartesian axes $O(x, y, z)$. We choose x-axis in the direction of the propagation of waves so that all particles on a line parallel to y-axis are equally displaced. Therefore all the field quantities will be independent of y co-ordinate. The motion is supposed to take place in two dimensions (x, z). Here u, w are the displacements of a point in the x, z directions respectively. In linear generalized theory of thermoelasticity, the governing fields equations for the temperature $T(x, z, t)$ and the displacement vector $\mathbf{u}(x, z, t) = (u, 0, w)$ in the absence of the body forces and heat sources (Green and Lindsay, 1972) are given by

$$c_{11}u_{,xx} + c_{44}u_{,zz} + (c_{13} + c_{44})w_{,xz} - \rho\ddot{u} = \beta_1(T + \tau_1\dot{T})_{,x}, \tag{1}$$

$$(c_{13} + c_{44})u_{,xz} + c_{44}w_{,xx} + c_{33}w_{,zz} - \rho\ddot{w} = \beta_3(T + \tau_1\dot{T})_{,z}, \tag{2}$$

$$K_1T_{,xx} + K_3T_{,zz} - \rho C_e(\dot{T} + \tau_0\ddot{T}) = T_0[\beta_1(\dot{u})_{,x} + \beta_3(\dot{w})_{,z}], \tag{3}$$

where

$$\beta_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3; \quad \beta_3 = 2c_{11}\alpha_1 + c_{13}\alpha_3; \tag{4}$$

c_{ij} are the elastic parameters; ρ is the density of the medium; C_e, τ_0 and τ_1 are the specific heat at constant strain, thermal and thermo-mechanical relaxations times respectively; K_1, K_3 and α_1, α_3 are respectively the coefficients of thermal conductivities and linear thermal expansions along and perpendicular to the axis of symmetry. The comma notation is used for spatial derivatives and superposed dot denotes time differentiation.

We define the following dimensionless quantities

$$x^* = \frac{v_1}{k_1}x, \quad z^* = \frac{v_1}{k_1}z, \quad t^* = \frac{v_1^2}{k_1}t, \quad u^* = \frac{v_1^3\rho}{k_1\beta_1 T_0}u, \quad w^* = \frac{v_1^3\rho}{k_1\beta_1 T_0}w, \quad T = \frac{T}{T_0},$$

$$\tau_0^* = \frac{v_1^2}{k_1}\tau_0, \quad \tau_1^* = \frac{v_1^2}{k_1}\tau_1, \quad c_1 = \frac{c_{33}}{c_{11}}, \quad c_2 = \frac{c_{44}}{2c_{11}}, \quad c_3 = \frac{c_{13} + \frac{1}{2}c_{44}}{c_{11}},$$

$$\bar{K} = K_3/K_1, \quad \bar{\beta} = \beta_3/\beta_1, \quad \varepsilon_1 = \frac{\beta_1^2 T_0}{\rho C_e v_1^2}, \quad \zeta^2 = \frac{\rho c^2}{c_{11}}, \tag{5}$$

where $v_1 = (c_{11}/\rho)^{1/2}$ is the velocity of compressional waves and $k_1 = K_1/\rho C_e$ is the thermal diffusivity in the x-direction. Here ε_1 is the thermoelastic coupling constant and τ_0^*, τ_1^* are the thermal relaxations constant.

Introducing the above quantities (5) in Eqs. (1)–(3), after suppressing the * and using superposed dot for time differentiation, we obtain

$$u_{,xx} + c_2 u_{,zz} + c_3 w_{,xz} - \ddot{u} = (T + \tau_1\dot{T})_{,x}, \tag{6}$$

$$c_3 u_{,xz} + c_2 w_{,xx} + c_1 w_{,zz} - \ddot{w} = \bar{\beta}(T + \tau_1\dot{T})_{,z}, \tag{7}$$

$$T_{,xx} + \bar{K}T_{,zz} - (\dot{T} + \tau_0\ddot{T}) = \varepsilon_1[(\dot{u})_{,x} + \bar{\beta}(\dot{w})_{,z}]. \tag{8}$$

The stresses and temperature gradient relevant to our problem in the plate are

$$\tau_{zz} = [(c_3 - c_2)u_{,x} + c_1 w_{,z} - \bar{\beta}(T + \tau_1\dot{T})]\beta_1 T_0, \tag{9}$$

$$\tau_{zx} = \beta_1 T_0 c_2(u_{,z} + w_{,x}), \tag{10}$$

$$T_{,z} = \frac{\partial T}{\partial z}. \tag{11}$$

For a plane harmonic wave traveling in the x-direction, the solutions u, w, and T of Eqs. (6)–(8) take the form

$$u = f(z)\exp[\mathrm{i}\xi(x - ct)], \tag{12}$$

$$w = g(z)\exp[\mathrm{i}\xi(x - ct)], \tag{13}$$

$$T = h(z)\exp[\mathrm{i}\xi(x - ct)], \tag{14}$$

where $c(= \omega/\xi)$ and ξ are phase velocity and wave number respectively; ω is the circular frequency, and $\mathrm{i} = \sqrt{-1}$. Substituting for u, w, and T from Eqs. (12)–(14) into Eqs. (6)–(8), we get

$$(c_2 D^2 - \xi^2 + \xi^2 c^2)f + \mathrm{i}\xi c_3 Dg - \xi^2 c\tau_G h = 0, \tag{15a}$$

$$\mathrm{i}\xi c_3 Df + (c_1 D^2 - c_2\xi^2 + \xi^2 c^2)g + \mathrm{i}\xi c\tau_G\bar{\beta}Dh = 0, \tag{15b}$$

$$\xi^2\varepsilon_1 cf - \mathrm{i}\xi c\varepsilon_1\bar{\beta}Dg - (\bar{K}D^2 - \xi^2 + \tau\xi^2 c^2)h = 0, \tag{15c}$$

where

$$D=\frac{\mathrm{d}}{\mathrm{d}z},\quad \tau=\tau_0+\frac{\mathrm{i}}{\xi c},\quad \tau_G=\tau_1+\frac{\mathrm{i}}{\xi c}. \tag{15d}$$

The solution to Eqs. (15a)–(15d) is

$$f(z)=P_1\exp(-\xi s_1 z)+P_2\exp(-\xi s_2 z)+P_2\exp(-\xi s_3 z)+Q_1\exp(\xi s_1 z)+Q_2\exp(\xi s_2 z)+Q_3\exp(\xi s_3 z), \tag{16a}$$

$$g(z)=m_1P_1\exp(-\xi s_1 z)+m_2P_2\exp(-\xi s_2 z)+m_3P_3\exp(-\xi s_3 z)-m_1Q_1\exp(\xi s_1 z)-m_2Q_2\exp(\xi s_2 z)-m_3Q_3\exp(\xi s_3 z), \tag{16b}$$

$$h(z)=\xi[l_1P_1\exp(-\xi s_1 z)+l_2P_2\exp(-\xi s_2 z)+l_3P_3\exp(-\xi s_3 z)+l_1Q_1\exp(\xi s_1 z)+l_2Q_2\exp(\xi s_2 z)+l_3Q_3\exp(\xi s_3 z)], \tag{16c}$$

where

$$m_j=\frac{[\bar{\beta}(c_2s_j^2+c^2-1)+c_3]s_j}{[s_j^2(c_3\bar{\beta}-c_1)+c_2-c^2]\mathrm{i}}, \tag{17a}$$

$$l_j=\frac{[(c_1s_j^2+c^2-1)-\mathrm{i}c_3s_jm_j]}{\xi c\tau_G}, \tag{17b}$$

and P_j, Q_j $(j=1,2,3)$ are arbitrary constants, and s_1, s_2, and s_3 are the roots of the equation

$$s^6+A_1s^4+A_2s^2+A_3=0, \tag{18}$$

where

$$A_1=-[\bar{K}c_2(c_2-c^2)+c_1c_2(1-\tau c^2)+\bar{K}c_1(1-c^2)-\bar{K}c_3^2-\varepsilon_1c_2\bar{\beta}^2\tau_Gc^2]/\Delta,$$

$$A_2=[\{\bar{K}(c_2-c^2)+c_1(1-\tau c^2)-\varepsilon_1\tau_G\bar{\beta}^2c^2\}(1-c^2)-c_3^2(1-\tau c^2)+c_2(c_2-c^2)(1-\tau c^2)+\varepsilon_1\{2c_3\bar{\beta}\tau_Gc^2-c_1\tau_Gc^2\}]/\Delta,$$

$$A_3=[-(c_2-c^2)[(1-\tau c^2)(1-c^2)-\varepsilon_1\tau_Gc^2]]/\Delta,$$

$$\Delta=\bar{K}c_1c_2.$$

The displacements and temperature to the plate are thus

$$u=[P_1\exp(-\xi s_1 z)+P_2\exp(-\xi s_2 z)+P_3\exp(-\xi s_3 z)+Q_1\exp(\xi s_1 z)+Q_2\exp(\xi s_2 z)+Q_3\exp(\xi s_3 z)]\exp[\mathrm{i}\xi(x-ct)], \tag{19a}$$

$$w=[m_1P_1\exp(-\xi s_1 z)+m_2P_2\exp(-\xi s_2 z)+m_3P_3\exp(-\xi s_3 z)-m_1Q_1\exp(\xi s_1 z)-m_2Q_2\exp(\xi s_2 z)-m_3Q_3\exp(\xi s_3 z)]\exp[\mathrm{i}\xi(x-ct)], \tag{19b}$$

$$T=\xi[l_1P_1\exp(-\xi s_1 z)+l_2P_2\exp(-\xi s_2 z)+l_3P_3\exp(-\xi s_3 z)+l_1Q_1\exp(\xi s_1 z)+l_2Q_2\exp(\xi s_2 z)+l_3Q_3\exp(\xi s_3 z)]\exp[\mathrm{i}\xi(x-ct)]. \tag{19c}$$

3. Boundary conditions

The boundary conditions are that stresses and temperature gradient on the surfaces of the plate should vanish. Hence for all x and t,

$$\tau_{zz} = \tau_{xz} = T_{,z} = 0 \quad \text{on } z = \pm d. \tag{20}$$

Making use of the boundary conditions (20) and bearing in mind the relations (9), (10), (11), and (19), we obtain a system of six algebraic equations involving the arbitrary constants P_1, P_2, P_3, Q_1, Q_2, and Q_3:

$$\sum_{j=1}^{3} (\mathrm{i}F - c_1 m_j s_j - \bar{\beta} l_j)(P_j \mathrm{e}^{-\xi s_j d} + Q_j \mathrm{e}^{\xi s_j d}) = 0, \tag{21a}$$

$$\sum_{j=1}^{3} (\mathrm{i}m_j - s_j)(P_j \mathrm{e}^{-\xi s_j d} - Q_j \mathrm{e}^{\xi s_j d}) = 0, \tag{21b}$$

$$\sum_{j=1}^{3} (-l_j s_j)(P_j \mathrm{e}^{-\xi s_j d} - Q_j \mathrm{e}^{\xi s_j d}) = 0, \tag{21c}$$

$$\sum_{j=1}^{3} (\mathrm{i}F - c_1 m_j s_j - \bar{\beta} l_j)(P_j \mathrm{e}^{\xi s_j d} + Q_j \mathrm{e}^{-\xi s_j d}) = 0, \tag{21d}$$

$$\sum_{j=1}^{3} (\mathrm{i}m_j - s_j)(P_j \mathrm{e}^{\xi s_j d} - Q_j \mathrm{e}^{-\xi s_j d}) = 0, \tag{21e}$$

$$\sum_{j=1}^{3} (-l_j s_j)(P_j \mathrm{e}^{\xi s_j d} - Q_j \mathrm{e}^{-\xi s_j d}) = 0, \tag{21f}$$

where $F = c_3 - c_2$.

4. Frequency equation

In order that the six boundary conditions be satisfied simultaneously the determinant of the coefficients of the arbitrary constants in Eqs. (21a)–(21f) must vanish. This gives an equation for the frequency of the plate oscillations. The frequency equation is found to factorize into two factors, each of which yields the equations

$$D_1 G_1 \coth(\xi s_1 d) - D_2 G_2 \coth(\xi s_2 d) + D_3 G_3 \coth(\xi s_3 d) = 0 \tag{22a}$$

and

$$D_1 G_1 \tanh(\xi s_1 d) - D_2 G_2 \tanh(\xi s_2 d) + D_3 G_3 \tanh(\xi s_3 d) = 0, \tag{22b}$$

where

$$D_j = \mathrm{i}F - c_1 m_j s_j - \bar{\beta} l_j, \tag{23}$$

$$G_1 = Y_2 Z_3 - Y_3 Z_2, \quad G_2 = Y_1 Z_3 - Y_3 Z_1, \quad G_3 = Y_1 Z_2 - Y_2 Z_1, \tag{24}$$

$$Y_j = \mathrm{i}m_j - s_j, \quad Z_j = -l_j s_j, \quad j = 1,2,3, \tag{25}$$

m_j and l_j are given in Eqs. (17a) and (17b).

These are the period equations which correspond to the symmetric and antisymmetric motion of the plate with respect to the medial plane $z = 0$. It can be shown that Eq. (22a) corresponds to the symmetric motion and Eq. (22b) corresponds to the antisymmetric motion.

The predictions from various theories can be obtained from Eqs. (22a) and (22b) as special cases:

- If $\tau = \tau_G = 0$, Eqs. (22a) and (22b) become the frequency equations of classical coupled thermoelasticity.
- If $\tau = \tau_G \neq 0$, Eqs. (22a) and (22b) become the frequency equations of the LS theory of generalized thermoelasticity Verma and Hasebe (in press).
- If $\varepsilon_1 = 0$, Eqs. (22a) and (22b) become the frequency equations of uncoupled generalized thermoelasticity Abubakar (1962).

4.1. Isotropic case

If we take

$$c_{11} = c_{33} = \lambda + 2\mu, \quad c_{44} = \mu, \quad K_1 = K_3 = K, \tag{26}$$

$$\alpha_1 = \alpha_3 = \alpha_t, \quad \beta_1 = \beta_3 = (3\lambda + 2\mu)\alpha_t, \tag{27}$$

the above Eqs. (22a) and (22b) reduce to the corresponding forms for an isotropic body with Lamé's parameters λ, μ; thermal conductivity K, and the coefficient of linear thermal expansion α_t. In this case, if $\tau = \tau_G \neq 0$, Eqs. (22a) and (22b) become the frequency equations of LS theory of generalized thermoelasticity, which has been discussed by Verma and Hasebe (1999). Massalas and Kalpakidis (1987a) have derived and discussed the frequency equation for thin isotropic plate of infinite length in the context of LS theory. They have studied the frequency equations under mixed boundary conditions and for isothermal and insulated edges. Further, with proper choice of parameters and boundary conditions, frequency Eqs. (22a) and (22b) agree with those obtained by Massalas and Kalpakidis (1987a) and Massalas (1986) (cf. Eqs. (26) and (27)) for symmetrical and antisymmetrical motions respectively, about the plane of symmetry of the plate. The phase velocity and attenuation constant (damping coefficient) of quasi-thermal mode (Fig. 1, mode 2) when $\xi \to \infty$, approach finite values, which is in agreement with Massalas and Kalpakidis (1987a) in the case of mixed boundary conditions and for insulated edges, instead of infinite ones predicted by coupled thermoelasticity Deresiewicz (1957). In the first mode of antisymmetric motions (in Fig. 1), the phase velocity increases monotonically with increasing wave number values ξ from $c = 0$ at $\xi = 0$ to $c = c_{\mathrm{R}}$ (Rayleigh waves) at $\xi = \infty$, which is characteristic of flexural waves. The results obtained for flexural mode (first mode) are in agreement with the corresponding results obtained by Ewing et al. (1957).

The discussion of transcendental Eqs. (22a) and (22b) in general is difficult; we therefore, consider the results for some limiting cases.

5. Symmetric modes

For waves long compared with the thickness *2d* of the plate, ξd is small and consequently ξds_1, ξds_2 and ξds_3 may be assumed small as long as c is finite. Hence the hyperbolic tangent functions can be replaced by their arguments and from Eq. (22b) we then obtain

$$(s_2^2 - s_3^2)(s_1^2 - s_3^2)(s_1^2 - s_2^2)[F_{11}H_2A_3 - F_{33}H_3 - F_{22}H_1A_3 - F_{11}H_3A_2 - F_{22}H_3A_1] = 0, \tag{28}$$

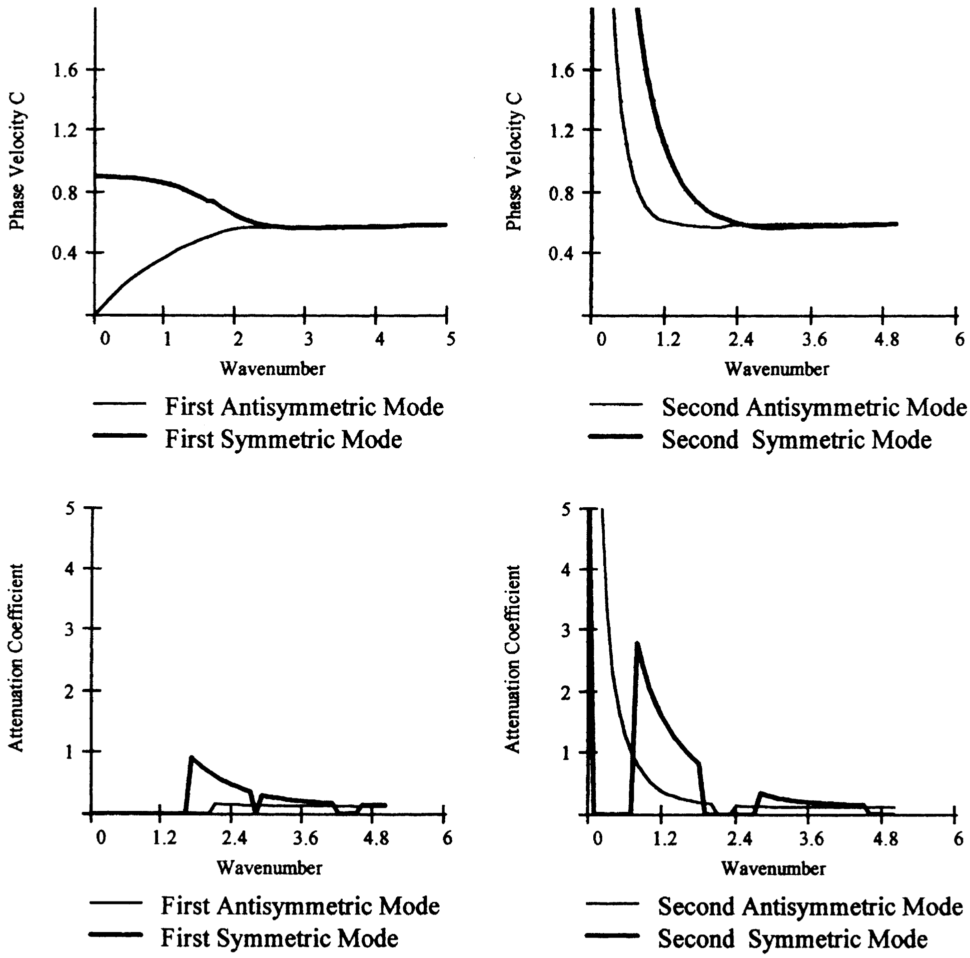


Fig. 1. Dispersion curves for antisymmetric and symmetric modes for aluminum material.

where

$$F_{11} = (c_3\bar{\beta} - c_1)c_2\bar{\beta}(c_2 - c_3) - c_2^2\bar{\beta}^2, \tag{29}$$

$$F_{22} = (c_3\bar{\beta} - c_1)(c^2 - 1)c_2 - c_2c_3\bar{\beta}(1 + c^2 - c_2) - (c^2 - 1)c_2\bar{\beta}^2, \tag{30}$$

$$F_{33} = (c^2 - 1)^2\bar{\beta}[(c_3\bar{\beta} - c_1) - \bar{\beta}] - 2c_3\bar{\beta}(c^2 - 1) - [((c^2 - 1)\bar{\beta} + c_3)(c^2 - c_2)], \tag{31}$$

$$H_1 = c_2\bar{\beta}(c_3\bar{\beta} - c_1), \tag{32}$$

$$H_2 = [(c_3\bar{\beta} - c_1)\{\bar{\beta}(c^2 - 1) + c_2(1 - \bar{\beta})\}], \tag{33}$$

$$H_3 = \bar{\beta}(c^2 - 1)[(c_3\bar{\beta} - c_1) - (c_2 - c^2)] - (c_3 - c_2)(c^2 - c_2). \tag{34}$$

Hence Eq. (28) is either

$$(s_2^2 - s_3^2)(s_1^2 - s_3^2)(s_1^2 - s_2^2) = 0 \tag{35a}$$

or

$$F_{11}H_2A_3 - F_{33}H_3 - F_{22}H_1A_3 - F_{11}H_3A_2 - F_{22}H_3A_1 = 0. \tag{35b}$$

If $s_1^2 = s_2^2$, $s_2^2 = s_3^2$, $s_3^2 = s_1^2$ the form of the original solution assumed, Eqs. (19a)–(19c) cannot satisfy the boundary conditions. Hence Eq. (35b) holds. This equation gives the phase velocity of long compressional or plate waves in generalized theory of thermoelasticity.

On using the isotropic relations Eqs. (26) and (27), expression (35b) reduces to

$$\left[2 - \frac{c^2}{c_2}\right]^2 [1 - c^2(\tau + \varepsilon_1\tau_G)] = 4[(c^2\tau - 1)(c^2 - 1) - \varepsilon_1 c^2\tau_G]. \tag{36}$$

Eq. (36) gives the phase velocity of long compressional or plate waves in generalized theory of thermoelasticity. In the framework of the classical elasticity ($\varepsilon_1 = 0$), Eq. (36) reduces to

$$c^2 = 4\beta^2\left(1 - \frac{\beta^2}{\alpha^2}\right), \tag{37}$$

which agrees with Ewing et al. (1957).

For very short waves and c such that s_1, s_2 and s_3 are real, ξd is large and the hyperbolic functions tend to unity. Hence Eq. (22b) becomes

$$\begin{aligned}&(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)[(s_1 + s_2 + s_3)(F_{11}H_3A_2 - F_{11}H_2A_3 + F_{22}H_1A_3 + F_{22}H_3A_1 + F_{33}H_3)\\ &\quad + s_1s_2s_3\{(s_1s_2 + s_2s_3 + s_3s_1)(F_{11}H_3 + F_{22}H_2 - F_{33}H_1) + (F_{11}H_1A_3 + F_{22}H_1A_2 + F_{22}H_2A_2\\ &\quad + F_{22}H_3 + F_{33}H_2)\}] = 0.\end{aligned} \tag{38a}$$

Evidently $(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)$ is a factor. Therefore, from Eq. (38a) we obtain

$$\begin{aligned}&(s_1 + s_2 + s_3)(F_{11}H_3A_2 - F_{11}H_2A_3 + F_{22}H_1A_3 + F_{22}H_3A_1 + F_{33}H_3) + s_1s_2s_3\{(s_1s_2 + s_2s_3 + s_3s_1)\\ &\quad \times (F_{11}H_3 + F_{22}H_2 - F_{33}H_1) + (F_{11}H_1A_3 + F_{22}H_1A_2 + F_{22}H_2A_2 + F_{22}H_3 + F_{33}H_2)\} = 0,\end{aligned} \tag{38b}$$

where s_1, s_2 and s_3 are roots of Eq. (18).

Eq. (38b) can be identified as the phase velocity equation for Rayleigh waves in transversely isotropic half-space.

On using the isotropic relations (26) and (27) expression (38b) becomes

$$[-(1 + s_3^2)^2\{s_1^2 + s_1^2 + s_1s_2 + c^2 - 1\} + 4s_1s_2s_3(s_1 + s_2)] = 0. \tag{39a}$$

Eq. (39a) can be identified as the phase velocity equation for Rayleigh waves in isotropic half-space. This is in agreement with the corresponding result of Nayfeh and Nasser (1971). In the framework of the classical elasticity ($\varepsilon_1 = 0$), Eq. (39a) reduces to

$$\left[2 - \frac{c^2}{c_2}\right]^4 = 16(1 - c^2)\left(1 - \frac{c^2}{c_2}\right). \tag{39b}$$

This is in agreement with the corresponding result of Nayfeh and Nasser (1971).

6. Antisymmetric modes

For waves long compared with thickness of the plate, s_1, s_2, and s_3 are real, we may replace the hyperbolic functions by the approximation

$$\tanh x \cong x - x^3/3.$$

After some algebraic transformation and reductions, and neglecting $\mathrm{O}[\xi d]^3$ we obtain

$$(s_1^2 - s_2^2)(s_2^2 - s_3^2)(s_3^2 - s_1^2)[(F_{33}H_1 - F_{22}H_2 - F_{11}H_3) - \frac{\gamma^2}{3}(F_{11}H_1A_3 + F_{33}H_2 + F_{22}H_1A_2 + F_{33}H_1A_1 + F_{22}H_3)] = 0. \tag{40}$$

Hence either

$$(s_1^2 - s_2^2)(s_2^2 - s_3^2)(s_3^2 - s_1^2) = 0 \tag{41a}$$

or

$$(F_{33}H_1 - F_{22}H_2 - F_{11}H_3) - \frac{\gamma^2}{3}(F_{11}H_1A_3 + F_{33}H_2 + F_{22}H_1A_2 + F_{33}H_1A_1 + F_{22}H_3) = 0, \tag{41b}$$

where $\gamma = \xi d$.

Eq. (41a) cannot satisfy the boundary conditions. Hence Eq. (41b) holds. On solving Eq. (41b) for a crystal of zinc, we note that the phase velocity decreases as the wavelength increases (Fig. 2). Therefore the Eq. (41b) is the dispersion equation of long flexural waves in generalized thermoelasticity.

Using the isotropic relations (26) and (27) Eq. (41b) reduces to

$$\frac{c^2}{c_2} - \frac{4\xi^2 d^2}{3}\left[(c_2 - 1)\left(1 + \frac{c^2}{c_2}\right) - \frac{c^2}{4c_2}(c^2 - 1)\right]. \tag{41c}$$

which agrees with Verma and Hasebe (1999).

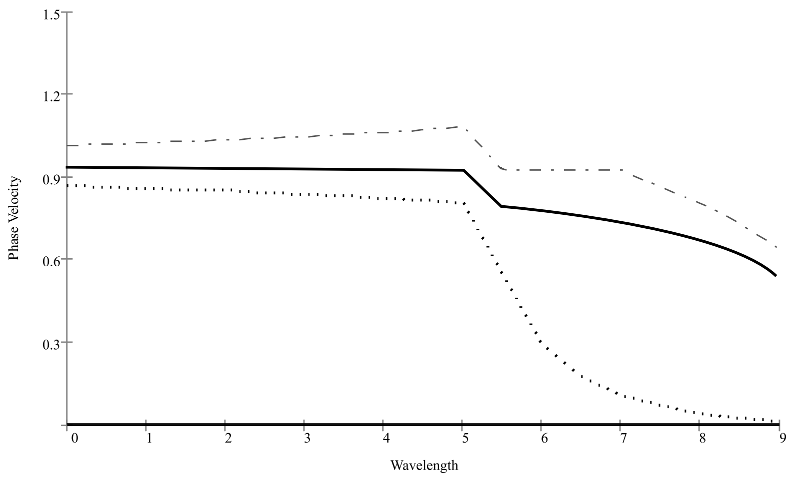


Fig. 2. Dispersion curves for antisymmetric modes of long flexural waves for a crystal of zinc (—: mode 1, - - -: mode 2, —: mode 3 – · – ·: mode 4).

For waves short compared with the thickness of the plate, that is $\xi d \to \infty$ and c such that s_1, s_2, and s_3 are real, Eq. (22a) reduces to Rayleigh equation (39b) and the propagation degenerates to Rayleigh waves associated with both free surfaces of the plate in generalized thermoelasticity.

7. Thermoelastic surface waves speed determination

7.1. Wave propagates in an arbitrary direction

In order to have a surface wave, the roots s_i^2 $(i = 1, 2, 3)$, of Eq. (18) must be either negative (so the square roots are pure imaginary) or complex number; this ensures that superposition of partial waves has the property of exponential decay. There are two cases:

(i) s_i^2, $i = 1$, 2, 3 all are negative;
(ii) s_1^2 is negative $s_2^2 = s_3^{2*}$, are complex conjugates (* means complex conjugate).

For case (i), as $d \to \infty$, $\{\tanh(\xi s_j d)\}^{\pm 1} \to (\pm 1)$, so Eqs. (22a) and (22b) reduces to

$$D_1 G_1 - D_2 G_2 + D_3 G_3 = 0. \tag{42}$$

For case (ii), $d \to \infty$, $\{\tanh(\xi s_1 d)\}^{\pm 1} \to \pm 1$ and if $s_2^2 = p + iq$, $s_3^2 = p - iq$, $(q > 0)$ then $\{\tanh(\xi s_2 d)\}^{\pm} \to \pm 1$ and $\{\tanh(\xi s_3 d)\}^{\pm 1} \to -(\pm 1)$, so, we have from Eqs. (22a) and (22b)

$$D_1 G_1 - D_2 G_2 - D_3 G_3 = 0. \tag{43}$$

Eqs. (42) and (43) can be solved for the thermoelastic surface wave velocity in the context of generalized thermoelasticity.

7.2. Wave propagation in principal direction (say x direction)

Similar to the situation described in Section 7.1, we have two cases:

(i) s_i^2, $i = 2$, 3 are negative;
(ii) $s_2^2 = s_3^{2^*}$, are complex conjugates.

Eq. (22a) and (22b) become Eq. (42), which can be simplified to find the thermoelastic surface waves.

7.2.1. Classical case

This case corresponds to the situation when the strain and temperature fields are not coupled with each other. In this case the thermo-mechanical coupling constant ε_1 is identically zero. Eq. (18) reduces to

$$s^2 \bar{K} + c^2 \tau - 1 = 0 \tag{44a}$$

and

$$c_1 c_2 s^4 + (c_3^2 - c_2^2 + c_2 c^2 + c_1 c^2 - c_1) s^2 + (c_2 - c^2)(1 - c^2) = 0. \tag{44b}$$

.

Eq. (44a) gives us $s^2 = [1 + c_2(\tau_0 + (i/\omega\xi))]/\bar{K}$, which defines the speed and the attenuation constant for the thermal wave. Clearly this is influenced by the thermal relaxation time τ_0.

Eq. (44b) is exactly the same equation which has been obtained and discussed by Abubakar (1962), which gives two period equations, for the symmetric and antisymmetric modes, respectively, for a free homogeneous transversely isotropic plate of thickness '2*d*'.

Further, if we define $V^2 = \omega^2/\zeta^2$ then Eqs. (42) and (43) with Eq. (44b) after some algebraic manipulations reduces to

$$[c_1 - (c_3 - c_2)^2 - c_1 V^2]^2 (c_2 - V^2) - c_1 c_2 V^4 (1 - V^2) = 0. \tag{45}$$

This equation is the same as Stoneley's (1943) phase velocity equation for Rayleigh waves in a transversely isotropic half-space, and reduces to the equation giving the velocity of Rayleigh waves in the isotropic case. Stoneley (1943) proved that this equation has only one real value for V^2 in the range

$$0 < V^2 < c_2. \tag{46}$$

For *thermoelastic isotropic material* in the context of generalized thermoelasticity, the Eqs. (42) and (43) reduces to

$$(1 - s_3^2)^2 \{s_1^2 + s_2^2 + s_1 s_2 + 1 - \zeta^2\} - 4 s_1 s_2 s_3 (s_1 + s_2) = 0. \tag{47}$$

Eq. (47) is the same as obtained and discussed by Nayfeh and Nasser (1971) and Sharma (1985), and Eq. (45) reduces to Eq. (39b).

This reveals that the elastic waves will be non-dispersive in this case, which is in agreement with Stoneley (1943) in the non-dimensional case.

7.2.2. *Case of coupled thermoelasticity*

This case corresponds to no thermal relaxation time, i.e. $\tau_0 = 0$ and hence $\tau = i/\omega$. In case isotropic materials, proceeding on the same lines, we again arrived at frequency equations of the form (39b). This is again in agreement with the corresponding result obtained by Chadwick (1960), Lockett (1985) and Sharma (1985). If we use the condition $\omega \ll 1$, then Eq. (45) reduces to

$$(1 + \varepsilon_1)\left(2 - \frac{\zeta^2}{c_2^2}\right)^4 = 16\{(1 + \varepsilon_1) - \zeta^2\}\left(1 - \frac{\zeta^2}{c_2^2}\right). \tag{48}$$

When $\varepsilon_1 = 0$, Eq. (48) becomes Eq. (47) and for $\varepsilon_1 \neq 0$ it corresponds to the result obtained by Sharma (1985).

8. Numerical discussion and conclusions

In general the waves are dispersive; the manner in which the long and short wavelength limits are connected requires a numerical solution of the Eqs. (22a) and (22b) is required. Moreover, the value of c which make s_1, s_2 and s_3 imaginary, the hyperbolic functions become periodic and so an infinite number of higher modes exists. Computation for the symmetric and antisymmetric modes have been carried out for a single crystal of zinc, for which, the basic physical data are, (Chadwick, 1960)

$$c_1 = 0.3851, \quad c_2 = 0.2365, \quad c_3 = 0.5485, \quad \bar{\beta} = 0.8991, \quad \bar{K} = 1, \quad \varepsilon_1 = 0.0221.$$

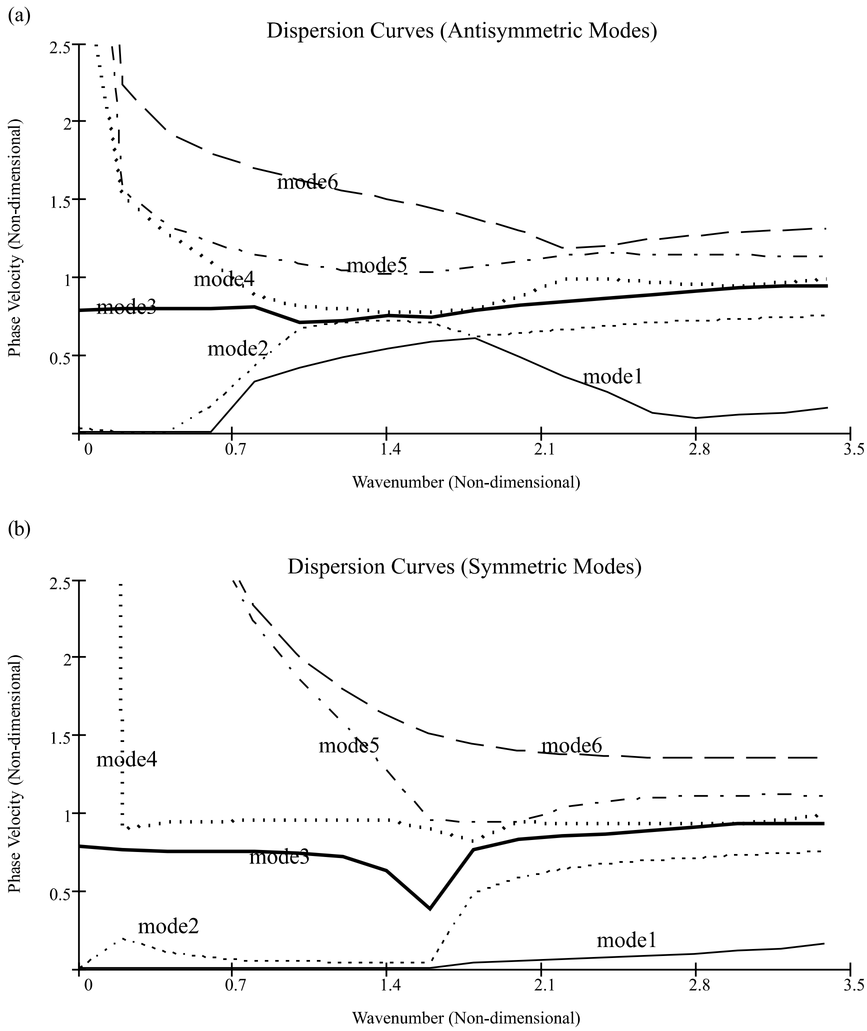


Fig. 3. Variation of phase velocity with wavenumber for the six lowest modes in GL theory of generalized thermoelasticity with $\tau_1 = 0.02$, $\tau_0 = 0.01$.

The results for the symmetric and antisymmetric vibrations (lowest six modes) are shown in Figs. 3–5 for different values of thermal–mechanical relaxation time τ_1, and thermal relaxations time τ_0

$$\tau_1 = 0.02, \quad \tau_0 = 0.01, \qquad \frac{\tau_1}{\tau_0} = 2 \quad \text{(Fig. 3)},$$
$$\tau_1 = 0.025, \quad \tau_0 = 0.02, \qquad \frac{\tau_1}{\tau_0} = 1.25 \quad \text{(Fig. 4)},$$
$$\tau_1 = 0.027, \quad \tau_0 = 0.025, \qquad \frac{\tau_1}{\tau_0} = 1.08 \quad \text{(Fig. 5)}.$$

Dispersion curves in the forms of variations of phase velocity (dimensionless) with wave number (dimensionless) are constructed at different values of thermal and thermo-mechanical relaxation times

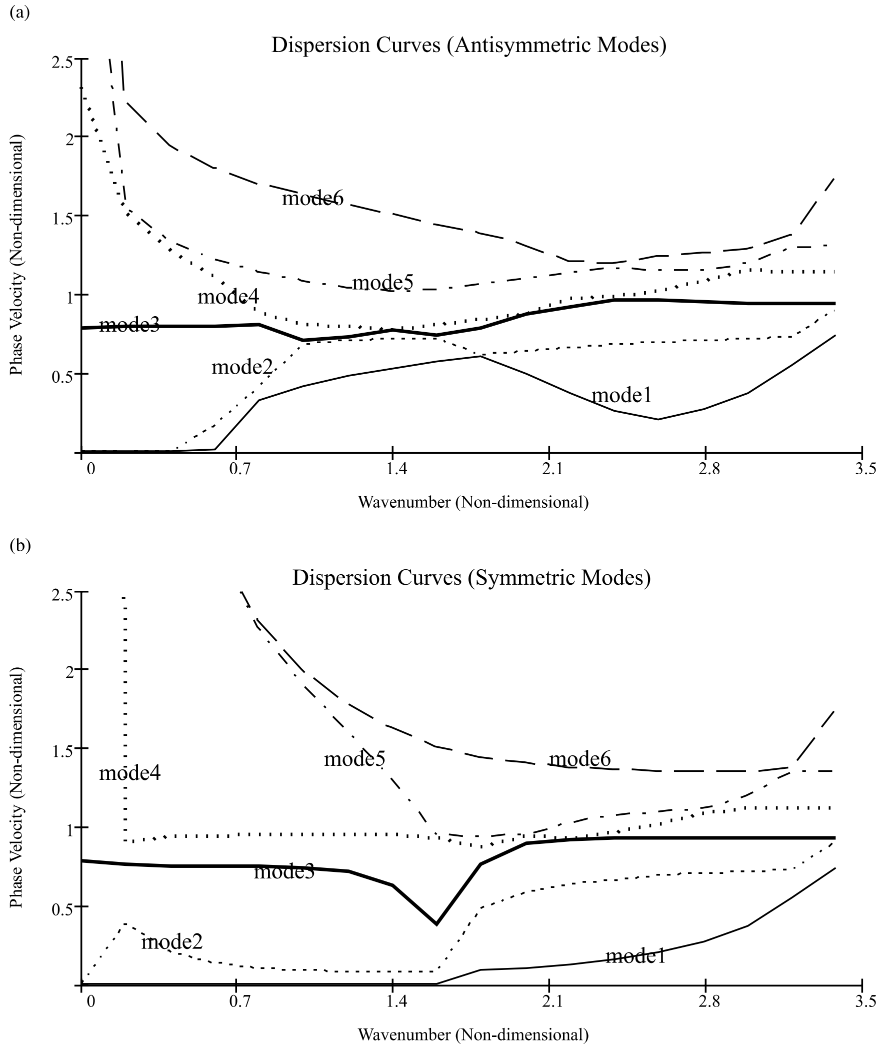


Fig. 4. Variation of phase velocity with wavenumber for the six lowest modes in GL theory of generalized thermoelasticity with $\tau_1 = 0.025$, $\tau_0 = 0.02$.

for crystal of zinc. At zero wave number limit, each figure display three wave speeds corresponding to quasi-longitudinal, quasi-transverse and quasi-thermal. It is obvious that the largest value corresponds to the quasi-longitudinal mode. As ξ increases, other higher modes appear in both cases (antisymmetric and symmetric). One of these seems to be associated with rapid change in the slope of the mode. Lower modes are found to influenced by the thermal relaxation times at low values of wave number both in symmetric and antisymmetric modes, while in higher modes, change is observed at high values of wave number. Figs. 3–5, show the variations of phase velocity with wave number for antisymmetric modes and Figs. 3–5 for symmetric modes. In Figs. 3–5, the phase velocity of mode 1 (antisymmetric) increases monotonically with increasing values of wave number from $C = 0$ at zero wave number limit to $C = C_R$ (Rayleigh wave speed, which is 0.428 (dimensionless) in this case) as $\xi \to \infty$, which is a characteristic flexural waves. Mode 2

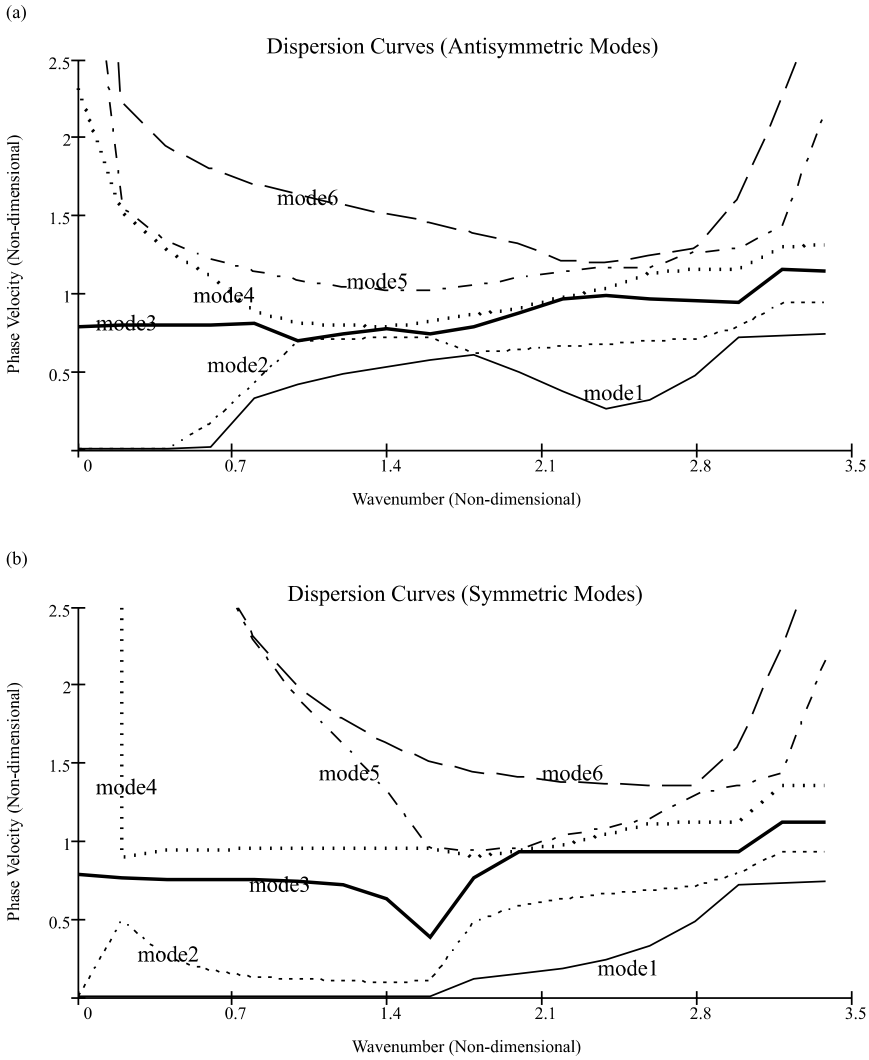
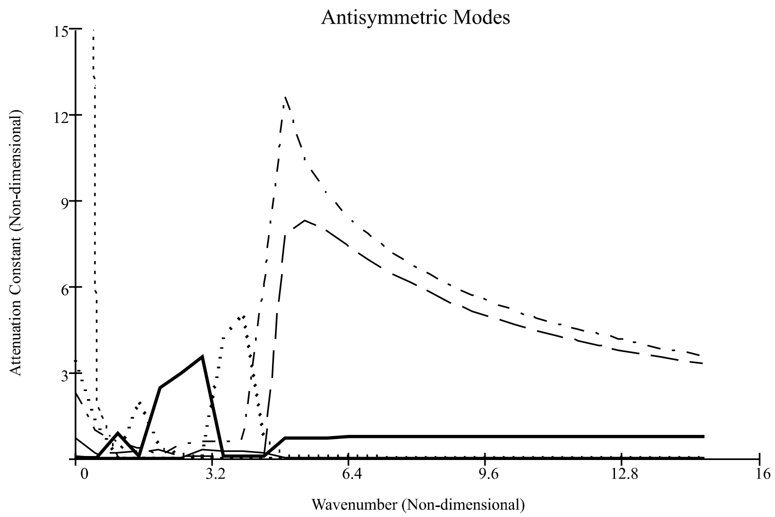


Fig. 5. Variation of phase velocity with wave number for the six lowest modes in GL theory of generalized thermoelasticity with $\tau_1 = 0.027$, $\tau_0 = 0.025$.

(antisymmetric), which is found to exist in thermoelasticity behaves like a mode 1, is a quasi-thermal (thermal mode) but having phase velocity higher than that of mode 1 and less than that of quasi-longitudinal (mode 3). The general shape of the dispersion curves of mode 1 (corresponding to first mode in elasticity) and mode 3 (corresponding to second mode in elasticity) are same as those obtained by Abubakar (1962).

Figs. 6–8 show the variation of attenuation constant (damping coefficients) with wave number for antisymmetric and symmetric waves for different ratio of the thermal relaxation times. Each of the curves in these figures corresponds to one of the branches in Figs. 3–5. It is observed from these figures that dissipation (damping) is high for small wave numbers and it dips down to local minima at certain value of the ratio τ_1/τ_0 (of the thermal relaxation times). From Figs. 3–5, it is apparent that attenuation constant for the

(a)



(b)

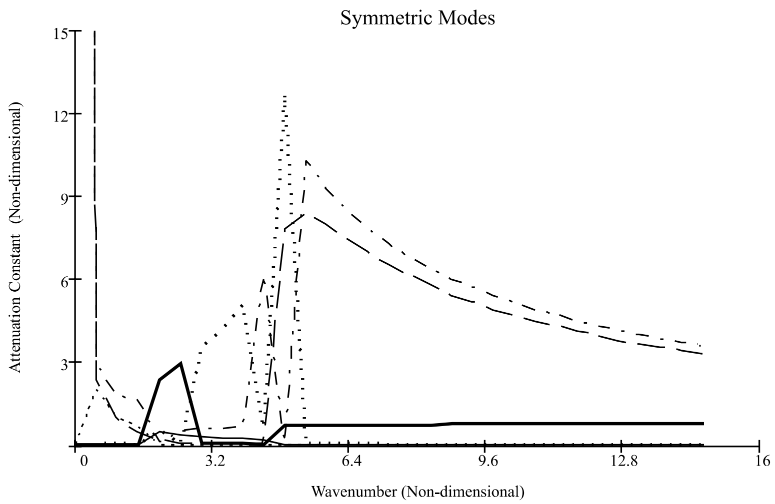
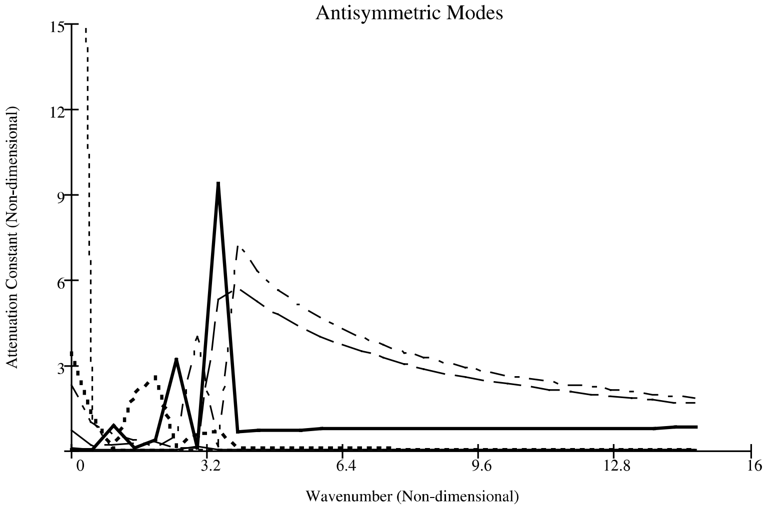


Fig. 6. Wave number dependence of thermoelastic attenuation constant in GL theory when $\tau_1 = 0.02$ and $\tau_0 = 0.01$ for the six lowest modes (—: mode 1, - - -: mode 2, **—**: mode 3, $\cdots$: mode 4, –·–·: mode 5, — — —: mode 6).

quasi-elastic modes increases as the ratio decreases while the attenuation constant for quasi-thermal mode increases, which agree with Tao and Prevost (1984).

The interaction of generalized thermoelastic waves in an infinite homogeneous transversely isotropic plate has been investigated in GL theory. Both the dispersion and the attenuation characteristics have been taken into consideration. The three motions namely, longitudinal, transverse and thermal of the medium are found dispersive and coupled with each other due to the thermal and anisotropic effects. The phase velocity and attenuation constant of the waves is get modified due to the thermal and anisotropic effects and is also influenced by the thermal relaxation time. Determination of thermoelastic surface waves

(a)



(b)

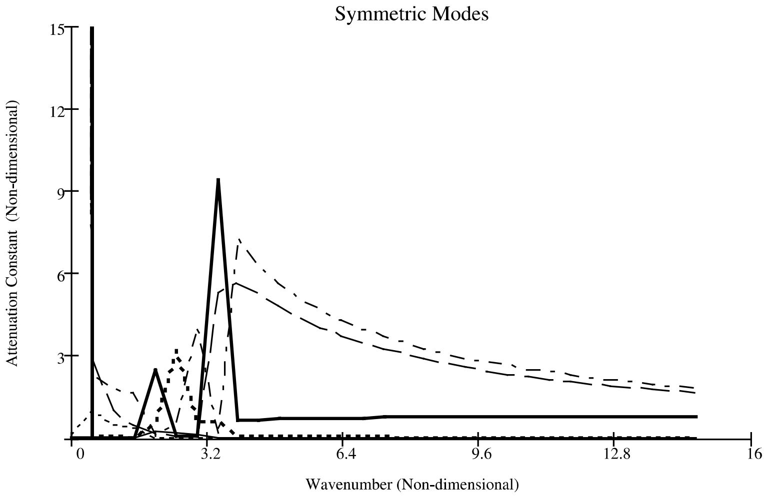


Fig. 7. Wave number dependence of thermoelastic attenuation constant in GL theory when $\tau_1 = 0.025$ and $\tau_0 = 0.02$ for the six lowest modes (—: mode 1, - - -: mode 2, **—**: mode 3, ···: mode 4, –·–·: mode 5, — — —: Mode 6).

speed is the byproduct of the analysis. Relevant results of previous investigations are deduced as special cases.

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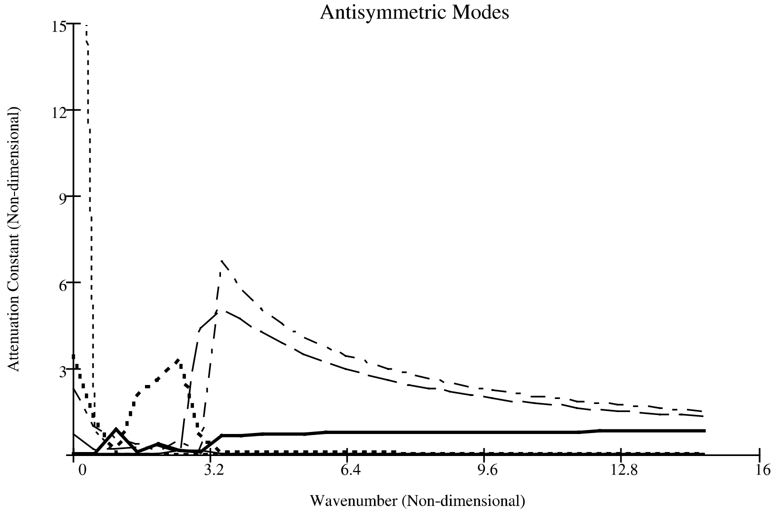
(a)



(b)

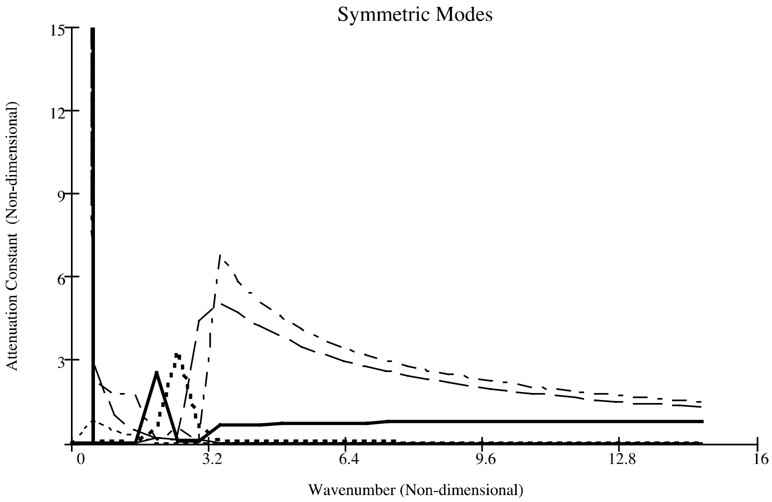


Fig. 8. Wave number dependence of thermoelastic attenuation constant in GL theory when $\tau_1 = 0.027$ and $\tau_0 = 0.025$ for the six lowest modes (—: mode 1, - - -: mode 2, **—**: mode 3, ···: mode 4, –·–·: mode 5, — — —: mode 6).

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